Assignment 1

1. Find the magnitude and direction of the following vectors:
   (a) $(-1,3)$,
   (b) $(0,-4,9)$,
   (c) $(1,0,5,-3)$.

2. Find all vectors that are perpendicular to the given ones:
   (a) $(-1,2)$,
   (b) $(2,4,-1)$,
   (c) $(-1,1,1,-4)$.

3. Find all vectors that are perpendicular to
   (a) $(3,2,1),(1,2,3)$,
   (b) $(6,0,-1),(2,0,0)$,
   (c) $(1,2,0,0),(0,1,-1,0),(0,1,0,1)$.

4. Find all vectors that are perpendicular to the triangle formed by
   (a) $(1,0,1),(1,0,-1),(0,1,1)$,
   (b) $(-2,3,1),(4,0,1),(0,-5,1)$.

5. Determine the angle between the vectors:
   (a) $(-1,3)$, $(6,3)$,
   (b) $(1,-1,3)$ $(-2,-1,-1)$,
   (c) $(2,3,0,0)$, $(7,-2,31,-109)$.

6.
   (a) Let $\mathbf{a},\mathbf{b},\mathbf{c}\in\mathbb{R}^n$. Show that the area of the triangle they form is given by
   $$A=\frac{1}{2}\sqrt{|\mathbf{b}-\mathbf{a}|^2|\mathbf{c}-\mathbf{a}|^2-|(\mathbf{b}-\mathbf{a})\cdot(\mathbf{c}-\mathbf{a})|^2}\ .$$
   (b) Find the area of the triangle with vertices at $(0,-1),(1,0),(-1,3)$.
   (c) Find the area of the triangle with vertices at $(0,2,0),(1,2,3)$, and $(1,0,0)$.

7. Let $\mathbf{u},\mathbf{v}$ be two vectors in $\mathbb{R}^n$. Verify that the vector
   $$\mathbf{w}=\frac{|\mathbf{v}|\mathbf{u}+|\mathbf{u}|\mathbf{v}}{|\mathbf{u}|+|\mathbf{v}|}\ ,$$
   bisects the angle between $\mathbf{u}$ and $\mathbf{v}$.

8. * Verify that the Euclidean distance satisfies the triangle inequality (the last axiom for a distance): For $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$,
   $$|\mathbf{x}-\mathbf{y}|\le|\mathbf{x}-\mathbf{z}|+|\mathbf{z}-\mathbf{y}|\ ,$$
   and show that the equality sign in this inequality holds if and only if $\mathbf{z}=\mathbf{x}+t(\mathbf{y}-\mathbf{x})$ for some $t\in[0,1]$. Hint: Use Cauchy-Schwarz Inequality.

9. *

 (a) Establish Lagrange's identity:

$$\left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{k=1}^{n} a_k^2 \sum_{j=1}^{n} b_j^2 - \frac{1}{2}\sum_{j,k=1}^{n} \left(a_j b_k - a_k b_j\right)^2 .$$

 (b) Deduce Cauchy-Schwarz Inequality from it.

10. * Prove the polarization identity in $\mathbb{R}^n$:

$$\mathbf{a} \cdot \mathbf{b} = \frac{1}{4}\left(|\mathbf{a}+\mathbf{b}|^2 - |\mathbf{a}-\mathbf{b}|^2\right) .$$

 What is its meaning?

11. * For $\mathbf{x} = (x_1, x_2), \mathbf{y} = (y_1, y_2)$, define

$$\|x - y\| = |x_1 - y_1| + |x_2 - y_2| .$$

 (a) Show that $\|\mathbf{x} - \mathbf{y}\|$ defines a distance on $\mathbb{R}^2$.

 (b) Show that there are points on the circle $x_1^2 + x_2^2 = 1$ whose distances to the origin as defined in (a) are different.

 (c) Draw the unit circle $\{(x_1, x_2) : \|(x_1, x_2)\| = 1\}$.

12. Describe the Euclidean motion

$$T\mathbf{x} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 2 \end{bmatrix} ,$$

 in terms of translation, rotation and reflection.

13. Find the Euclidean motion that fixes the origin and send $(-1, 1)$ to $(\sqrt{2}, 0)$.

14. Find the formula for the reflection with respect to the y-axis.

15. Find the formula for the reflection with respect to the straight line $x - 2y = 0$.

16. Determine the magnitude and direction of the following cross product:

 (a) $(0, -2, -9) \times (1, 2, -4)$,

 (b) $(0, -6, 1) \times (3, 0, 5)$,

 (c) $(1, -1, 2) \times (-2, 2, -4)$.

17. Consider the points $(1, 1, 0), (1, 0, 1)$ and $(0, 1, 1)$.

 (a) Find the area of the triangle with vertices at these points.

 (b) Find the volume of the parallelepiped with vertices at these points together with $(0, 0, 0)$.

18. Consider $(1, 3, -2), (2, 4, 5)$ and $(-3, -2, 2)$.

 (a) Find the area of the triangle with vertices at these points.

 (b) Find the volume of the parallelepiped with vertices at these points together with $(0, 0, 0)$.

19. Determine whether the following points are coplanar or not:

$$(1, 3, -2),\ (3, 4, 1),\ (2, 0, -2),\ (4, 8, 4)\ .$$

20. (a) Establish

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} .$$

(b) Show that

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})\ .$$

21.

(a) Let $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ be two non-zero vectors satisfying

$$u_2v_3 - u_3v_2 = 0,\ u_1v_3 - u_3v_1 = 0,\ u_1v_2 - u_2v_1 = 0\ .$$

Show that there is some non-zero α such that $\mathbf{v} = \alpha\mathbf{u}$.

(b) Show that two non-zero vectors in $\mathbb{R}^3$ satisfying $\mathbf{u} \times \mathbf{v} = 0$ if and if $\mathbf{v} = \alpha\mathbf{u}$ for some non-zero α.

22. * Verify the following properties for the cross product: For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$ and scalars α, β,

(a)

$$\alpha(\mathbf{u} \times \mathbf{v}) = (\alpha\mathbf{u}) \times \mathbf{v}\ ,$$

(b)

$$\mathbf{u} \times (\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha\mathbf{u} \times \mathbf{v} + \beta\mathbf{u} \times \mathbf{w}\ .$$

23. * For 3-vectors $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$, establish the following identities:

(a)

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = \mathbf{v}(\mathbf{u} \cdot \mathbf{w}) - \mathbf{w}(\mathbf{u} \cdot \mathbf{v})\ .$$

(b) (Jacobi's identity)

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) + \mathbf{v} \times (\mathbf{w} \times \mathbf{u}) + \mathbf{w} \times (\mathbf{u} \times \mathbf{v}) = 0\ .$$

24. Let P, Q, and R be three points in space. Denote by $\overline{PQ}$ the vector from Q to P etc. Show that the distance from R to $\overline{PQ}$ is given by the formula

$$d = \frac{|\overline{PR} \times \overline{QR}|}{|\overline{PQ}|}\ .$$

25. * Let P, Q, R be three points lying in a plane in $\mathbb{R}^3$ and S another point. Show that the distance from S to the plane is given by

$$d = \frac{|\overline{SP} \cdot (\overline{SQ} \times \overline{SR})|}{|\overline{QP} \times \overline{RP}|}\ .$$

Suggestion: Look at the volume of the pyramid formed by these four points. The volume of a pyramid is given by $\frac{1}{3}ha$ where h is its height and a its base area. It is equal to $\frac{1}{6}$ of the parallelepiped spanned by these vectors (taking S as the origin).

26. * Show that the area of the triangle with vertices at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is given by half of the absolute value of the determinant, that is,

$$\frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}.$$